

Review Exam one MTH 512 , Fall 2019

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QUESTION 1. Let A be a 3×5 such that $A \overbrace{2R_2} B \overbrace{-R_2 + R_3 \to R_3} D = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

(i) Find the solution set to the system $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}$ [Hint: Note that the solution set is a subset of R^5 and think!].

SOLUTION 1.1. We need to form the augmented matrix. Note that A is the coefficient matrix. Hence $[A| \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}]$ is the augmented matrix. By hypothesis A is reduced to D by row operations. Hence here we go

$[A| \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}] \overbrace{2R_2} [B| \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}] \overbrace{-R_2 + R_3 \to R_3} D = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 & | & -1 \\ 0 & 1 & 2 & 0 & 3 & | & 2 \\ 0 & 0 & 0 & 1 & 2 & | & 4 \end{bmatrix} \overbrace{R_3 + R_1 \to R_1} F = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & | & 3 \\ 0 & 1 & 2 & 0 & 3 & | & 2 \\ 0 & 0 & 0 & 1 & 2 & | & 4 \end{bmatrix}$. Hence we stop and read

$x_1 = 3 - 2x_3 - x_5, x_2 = 2 - 2x_3 - 3x_5, x_4 = 4 - 2x_5$. Note x_1, x_2, x_4 are leading variables and $x_3, x_5 \in R$ (free variables).

Thus the solution set = $\{(3 - 2x_3 - x_5,\ 2 - 2x_3 - 3x_5,\ x_3,\ 4 - 2x_5, x_5) \mid x_3, x_5 \in R\}$

Since the system is not homogeneous, the solution set is a SUBSET of R^5 but NEVER a subspace of R^5 and hence it cannot be written as span. Also; note that we cannot talk about independent number (dimension) [since it is not a Subspace].

(ii) Find Elementary matrices E_1, E_2 such that $E_1E_2A = D$

SOLUTION 1.2. By staring at the row operations from A to D and $E_1E_2 = D$, we see that the first row operation corresponds to E_2 and the second row operation corresponds to E_1. Hence $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

(iii) Let $D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$. Find the matrix D without doing the actual multiplication of these 5 matrices [Stare well and think!]

SOLUTION 1.3. By staring, we observe that the first 4 matrices are elementary matrices. Hence

$\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \overbrace{-2R_1 + R_2 \to R_2} \begin{bmatrix} 2 & 4 \\ 0 & -6 \end{bmatrix} \overbrace{2R_2} \begin{bmatrix} 2 & 4 \\ 0 & -12 \end{bmatrix} \overbrace{-R_2 + R_1 \to R_1}$

$\begin{bmatrix} 2 & 16 \\ 0 & -12 \end{bmatrix} \overbrace{2R_2 + R_1 \to R_1} \begin{bmatrix} 2 & -8 \\ 0 & -12 \end{bmatrix} = D$

QUESTION 2. (i) Let A be an $n \times n$ invertible matrix. Convince me (i.e. prove) that if a is an eiginvalue of A, then a^{-1} is an eigenvalue of A^{-1}. Also, convince me that $E_a = E_{a^{-1}}$.

SOLUTION 2.1. Since a is an eigenvalue of A and A is invertible, we conclude that $a \neq 0$ and there exists a nonzero point Q in R^n such that $AQ^T = aQ^T$. Multiply both sides with A^{-1}, we get $Q^T = aA^{-1}Q$. Thus $A^{-1}Q^T = \frac{1}{a}Q^T$. Thus $1/a$ is an eigenvalue of A^{-1}.

As we learned from Elementary Math, to show that two sets , say F, K, are equal, we need to show that $F \subseteq K$ and $K \subseteq F$.

Hence we need to show that $E_a \subseteq E_{a^{-1}}$ and $E_{a^{-1}} \subseteq E_a$.

So, let $Q \in E_a$. We show $Q \in E_{a^{-1}}$. Thus $AQ^T = aQ^T$. Multiply both sides with A^{-1}, we get $Q^T = aA^{-1}Q$. Thus $A^{-1}Q^T = \frac{1}{a}Q^T$. Thus $Q \in E_{a^{-1}}$. Hence $E_a \subseteq E_{a^{-1}}$.

Now let $W \in E_{a^{-1}}$. We show $W \in E_a$. Hence $A^{-1}W^T = \frac{1}{a}W^T$. Multiply both sides with A. Thus $W^T = \frac{1}{a}AW^T$. Hence $AW^T = aW^T$. Hence $W \in E_a$, and therefore $E_{a^{-1}} \subseteq E_a$. Since $E_a \subseteq E_{a^{-1}}$ and $E_{a^{-1}} \subseteq E_a$, we conclude that $E_{a^{-1}} = E_a$.

(ii) Given A is a 3×3 diagnolizable matrix with eigenvalues 2, -2 such that $E_{-2} = span\{(1, 2, 3), (-1, -2, -2)\}$ and $E_2 = span\{(-1, -1, -3)\}$.

a. Find $|A|$ and Trace(A)

SOLUTION 2.2. Since A is diagnolizable, by staring at E_{-2} and E_2 we conclude that 2 is repeated once and -2 is repeated twice. Hence |A| = (-2)(-2)(2) = 8. Trace(A) = -2 + -2 + 2 = -2.
NOTE that A is diagnolizable is not needed in this question! right?

b. Find a diagonal matrix D and an invertible matrix Q such that $D = QAQ^{-1}$ (Do not calculate Q^{-1}).

SOLUTION 2.3. As explained in class, many possibilities. For example: $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & -2 \\ 3 & -3 & -2 \end{bmatrix}$

c. Find $C_{A^{-1}}(\alpha)$.

SOLUTION 2.4. From question (2), we conclude that $\frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}$ are the eigenvalues of A^{-1}. Hence $C_{A^{-1}}(\alpha) = (\alpha + \frac{1}{2})^2(\alpha - \frac{1}{2})$.

d. Find C_{A^2} and calculate A^2.

SOLUTION 2.5. Let Q, D as in Solution 2.3. Hence $Q^{-1}DQ = A$. Thus $Q^{-1}D^2Q = A^2$. Stare at D^2. You observe that $D^2 = 4I_3$. Hence $4Q^{-1}I_3Q = A^2$. Hence $A^2 = 4I_3$. Thus $C_{A^2}(\alpha) = |\alpha I_3 - 4I_3| = (\alpha - 4)^3$.

(iii) Let A be an $n \times n$ matrix. Suppose that there is a real number r such that the sum of all numbers in each column of A equals r. Convince me that r is an eigenvalue of A.

SOLUTION 2.6. Consider the matrix A^T. Then the sum of all numbers in each row of A^T equals r. Hence $A^T \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. Then r is an eigenvalue of A^T. We know that A^T and A have the same eigenvalues. Thus r is an eigenvalue of A.

(iv) Let A be a 13×13 matrix. Convince me that A must have at least one real eigenvalue.

SOLUTION 2.7. Note that the degree of $C_A(\alpha)$ is 13. So we set $C_A(\alpha) = 0$. Common knowledge (public knowledge) every polynomial of odd degree must have at least one real root. Thus A must have at least one real eigenvalue.

(v) Let A be a 4×4 matrix and $C_A(\alpha) = (\alpha-3)^2(\alpha-2)^2$ such that $E_3 = span\{(2,1,1,1)\}$ and $E_2 = span\{(-2,1,0,1)\}$.

a. What is the solution set to the system $A\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = 5\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$?

SOLUTION 2.8. By staring at $C_A(\alpha)$. We conclude that 5 is not an eigenvalue of A. Hence the solution set is $\{(0,0,0,0)\}$.

b. Let $F = 5I_4 + 2A^{-1} + 3A$. Give me a nonzero point Q and a real number a such that $FQ^T = aQ^T$.

SOLUTION 2.9. Fist observe that A^{-1} exists, since $|A| = (2)(2)(3)(3) = 36 \neq 0$. Choose any nonzero point Q in E_2 or E_3. We Know from solution 2.1 that $Q \in E_{\frac{1}{2}}$ or $Q \in E_{\frac{1}{3}}$ (note $E_{\frac{1}{2}}$ and $E_{\frac{1}{3}}$ are eigenspaces of A^{-1}).
Let us choose $Q = (-2,1,0,1) \in E_2$. Then
$FQ^T = [5I_4 + 2A^{-1} + 3A]Q^T = 5I_4Q^T + 2A^{-1}Q^T + 3AQ^T = 5Q^T + 2(0.5Q^T) + 3(2Q^T) =$
$5Q^T + Q^T + 6Q^T = 12Q^T$ (i.e., 12 is an eigenvalue of F).

QUESTION 3. Let $A = \begin{bmatrix}-c_5 & a_2 & a_3 & -2c_1 & a_5\\ c_3 & b_2 & b_3 & -c_1 & b_5\\ c_1 & -2 & c_3 & -1 & c_5\end{bmatrix}$. Given A is row-equivalent to $B = \begin{bmatrix}2&4&4&2&4\\0&1&1&3&1\\0&0&0&0&0\end{bmatrix}$.

(a)Find the matrix A.

SOLUTION 3.1. Note A_i means the ith column of A and $_iA$ means the ith row of A

By staring, $Row(A) = span\{(2,4,4,2,4),(0,1,1,3,1)\}$. As explained, each row of A is a linear combination of (2, 4, 4, 2, 4), (0, 1, 1, 3, 1)

Hence $_3A = (c_1,-2,c_3,-1,c_5) = a(2,4,4,2,4) + b(0,1,1,3,1) = (2a, 4a+b, 4a+b, 2a+3b, 4a+b)$. Find a, b. Hence $4a + b = -2$ and $2a + 3b = -1$. Now solve! we get $a = -0.5$ and $b = 0$. Thus $_3A = (-1,-2,-2,-1,-2)$. Hence $c_1 = -1$, $c_3 = -2$, $c_5 = -2$.

Similarly $_2A = (-2,b_2,b_3,1,b_5) = a(2,4,4,2,4) + b(0,1,1,3,1) = (2a, 4a+b, 4a+b, 2a+3b, 4a+b)$. Find a, b. Hence $2a = -2$ and $2a + 3b = 1$. Now solve! we get $a = -1$ and $b = 1$. Thus $_2A = (-2,-3,-3,1,-3)$.

Similarly $_1A = (2,a_2,a_3,2,a_5) = a(2,4,4,2,4) + b(0,1,1,3,1) = (2a, 4a+b, 4a+b, 2a+3b, 4a+b)$. Find a, b. Hence $2a = 2$ and $2a + 3b = 2$. Now solve! we get $a = 1$ and $b = 0$. Thus $_1A = (2,4,4,2,4)$.

Hence $A = \begin{bmatrix}2&4&4&2&4\\-2&-3&-3&1&-3\\-1&-2&-2&-1&-2\end{bmatrix}$.

(b) Find a basis of $Col(A)$.

As explained, to find a basis for Col(A). We stare at B, we locate the columns in B that have the "leaders". Here we see that the leaders are located in B_1 and B_2. Thus we MUST choose A_1, A_2 from A to form a basis for $Col(A)$.

Hence a basis for Col(A) is $Badawi = \{(2,-2,-1),(4,-3,-2)\}$.

Hence $Col(A) = span\{(2,-2,-1),(4,-3,-2)\}$.

QUESTION 4. Given $B = \{(0,1,1),(1,0,-1),(2,-2,-1)\}$ is a basis for R^3 and $Q = (2,6,-1) \in R^3$. Find $[Q]_B$.

SOLUTION 4.1. Form a matrix P, 3×3, where each column of P is a point in B. Now you may solve the system $PX = Q^T$. Then the point in the solution set is $[Q]_B$. Another way, find P^{-1}. Then $P^{-1}Q^T = [Q]_B$.

QUESTION 5. Let $D = span\{(3a+5b+2,\ -2b+1,\ 6a+8b+5,\ 6b-3,\ 3a+3b+3) \mid a,b \in R\}$.

(a) Convince me that D is a subspace of R^5.

SOLUTION 5.1. As explained, D will be a subspace "if each coordinate can be written as linear combination of linear variables." There are many ways. For example: Let $w = 3a+5b+2, v = -2b+1$. Note that $w, v \in R$ (since a, b in R). Hence $6a+8b+5 = 2w+v, 6b-3 = -3v, 3a+3b+3 = w+v$.

Thus $D = span\{(w,\ v,\ 2w+v,\ -3v,\ w+v) \mid w,v \in R\}$. Hence $D = span\{(1,0,2,0,1),(0,1,1,-3,1)\}$

(b) Find an orthogonal basis of D.

SOLUTION 5.2. Just Use Gram Schmidt Method.

QUESTION 6. Let $A = \begin{bmatrix} 2 & 4 & 1 & -3 \\ -2 & b_2 & b_3 & b_4 \\ -2 & -4 & c_3 & c_4 \\ -2 & -4 & -1 & d_4 \end{bmatrix}$. Assume that a point $Q = (x_1, x_2, x_3, x_4)$ is selected randomly from R^4. Find all possible values of $b_2, b_3, b_4, c_3, c_4, d_4$ so that the system $A \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = Q^T$ has a unique solution.

SOLUTION 6.1. We know that the claim will be correct iff $|A| \neq 0$. So we set $|A| \neq 0$. So let us calculate $|A|$.

$$A = \begin{bmatrix} 2 & 4 & 1 & -3 \\ -2 & b_2 & b_3 & b_4 \\ -2 & -4 & c_3 & c_4 \\ -2 & -4 & -1 & d_4 \end{bmatrix} \overbrace{R_1 + R_2 \to R_2} \overbrace{R_1 + R_3 \to R_3} \overbrace{R_1 + R_4 \to R_4} B = \begin{bmatrix} 2 & 4 & 1 & -3 \\ 0 & b_2 + 4 & b_3 + 1 & b_4 - 3 \\ 0 & 0 & c_3 + 1 & c_4 - 3 \\ 0 & 0 & 0 & d_4 - 3 \end{bmatrix}.$$

Hence $|A| = |B| = 2(b_2 + 4)(c_3 + 1)(d_4 - 3)$.

Thus $|A| \neq 0$ if $b_2 \neq -4$, $c_3 \neq -1$, $d_4 \neq 3$, $b_3, b_4, c_4 \in R$.

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